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The M5-brane and non-commutative loop space

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Abstract

We investigate, in a certain decoupling limit, the effect of having a constant C-field on the M-theory 5-brane using an open-membrane probe. We define an open-membrane metric for the 5-brane that remains non-degenerate in the limit. The canonical quantization of the open-membrane boundary leads to a non-commutative loop space which is a functional analogue of the non-commutative geometry that occurs for D-branes.

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1. Introduction

Before considering the M2/M5 system it is instructive to first briefly review the relation between D-branes and non-commutative geometry [1]. Consider a fundamental string F1 ending on a Dp-brane via a point or 0-brane. The effective tensions τ of the string and the Dp-brane behave like $\tau_{\mathrm{F1}} \sim 1$, $\tau_{Dp} \sim 1/g_s$. Therefore, for small g_s, the string is much lighter than the Dp-brane and can be treated as a test string probing the Dp-brane. Furthermore, the effective gravitational couplings $G_N\tau$ (Newton's constant times tension) behave like $G_N\tau_{\mathrm{F1}} \sim g_s^2$, $G_N\tau_{Dp} \sim g_s$ and therefore we can assume that the spacetime background is approximately flat. The open-string action reads

$$S = \frac{1}{2\alpha'}\int_{M^2} \mathrm{d}^2\sigma\, \bar\partial X^M \partial X^N \eta_{MN} + \frac{1}{2}\int_{\partial M^2} \mathrm{d}\tau\, \mathcal{F}_{\mu\nu} X^\mu \dot X^\nu, \tag{1}$$

where $\mathcal{F}_{\mu\nu}$ is the constant background field strength on the Dp-brane. We assume that the only non-vanishing components of $\mathcal{F}$ are $\mathcal{F}_{r's'}$, where we have decomposed the worldvolume index μ as $\mu = (r, r')$ with $r = 0, 1, \ldots, p - \mathrm{rank}\,\mathcal{F}$ and $r' = p + 1 - \mathrm{rank}\,\mathcal{F}, \ldots, p$:

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_{rs} = 0 & 0 \\ 0 & \mathcal{F}_{r's'} \end{pmatrix}. \tag{2}$$

We consider now the following decoupling limit (see, e.g., [2]). We take $\epsilon \to 0$ such that[1]

$$\eta_{r's'} \sim \epsilon\, \eta_{r's'}, \qquad \alpha' \sim \epsilon^{1/2}\alpha', \tag{3}$$

[1] In (3) it is understood that the $\eta_{r's'}$ and α' occurring on the right-hand side are ϵ-independent.

while keeping all other quantities fixed. The open string action scales as follows:

$$S \sim \frac{1}{2\epsilon^{1/2}\alpha'}\int_{M^2} \mathrm{d}^2\sigma\ \bar{\partial}X^m\partial X^n\eta_{mn} + \frac{1}{2\epsilon^{1/2}\alpha'}\int_{M^2} \mathrm{d}^2\sigma\ \bar{\partial}X^r\partial X^s\eta_{rs}$$
$$+\frac{\epsilon^{1/2}}{2\alpha'}\int_{\Sigma} \mathrm{d}^2\sigma\ \bar{\partial}X^{r'}\partial X^{s'}\eta_{r's'} + \frac{1}{2}\int_{\partial M^2} \mathrm{d}\tau\ \mathcal{F}_{r's'}\,X^{r'}\dot{X}^{s'}. \tag{4}$$

One may now argue (see, e.g., [2, 3]) that the dynamics of the $F1/Dp$ system is dominated by the last so-called Wess–Zumino term, i.e.

$$S \sim \tfrac{1}{2}\int_{\partial M^2} \mathrm{d}\tau\ \mathcal{F}_{r's'}X^{r'}\dot{X}^{s'}. \tag{5}$$

Moreover, the open string metric is finite in this limit and is given by the maximal rank matrix

$$G_{\mu\nu} = \begin{cases} \eta_{\mu\nu} & \text{for} \quad \mu,\nu = r,s \\ -\alpha'^2\,\mathcal{F}_{\mu\rho}\,\eta^{\rho\sigma}\,\mathcal{F}_{\sigma\nu} & \text{for} \quad \mu,\nu = r',s'. \end{cases} \tag{6}$$

The equations of motion corresponding to the Wess–Zumino term read

$$\dot{X}^{r'} = 0, \tag{7}$$

i.e. there are additional Dirichlet conditions: the endpoint of the string is not allowed to move in the r' directions. The non-commutative nature of the D-brane arises from quantizing the Wess–Zumino term (5). Applying the standard canonical quantization procedure leads to the following non-zero Dirac brackets:

$$\{X^{r'}(\sigma), X^{s'}(\sigma')\} = (\mathcal{F}^{-1})^{r's'}\delta(\sigma-\sigma'). \tag{8}$$

We thus conclude that the string probing the Dp-brane sees a non-commutative geometry in the r' directions of the Dp-brane worldvolume.

2. The M2/M5 system

The M-theory origin of the $F1/Dp$ system is a M2/M5 system, i.e. an open membrane ending on a 5-brane in an 11-dimensional supergravity background. The membrane boundary is a string that is constrained to lie within the 5-brane. The action for the open bosonic membrane is as follows:

$$S = S_k + \int_{M^3} f_2^*C + \int_{\partial M^3} f_1^*b, \tag{9}$$

where the kinetic term can be written in Polyakov form as

$$S_k = \frac{1}{2(\ell_p)^2}\int_{M^3} \mathrm{d}^3\sigma\,\sqrt{-\det\gamma}\,\big(-\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\hat{g}_{MN} + \ell_p^2\big). \tag{10}$$

Here ℓ_p is the $D = 11$ Planck constant, $\hat{g}_{MN}$ is the $D = 11$ spacetime metric and $\gamma_{\alpha\beta}$ is the auxiliary worldvolume metric. The maps f_2 and f_1 denote the embedding of the membrane and its boundary into the spacetime and the 5-brane, respectively. The worldvolume 3-form f_2^*C is the pull-back of the $D = 11$ 3-form potential C to the membrane worldvolume and, similarly, f_1^*b is the pull-back of the 5-brane 2-form potential b to the boundary of the membrane. In terms of components, we write

$$(f_2^*C)_{\alpha\beta\gamma} = \partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P C_{MNP}, \qquad (f_1^*b)_{ij} = \partial_i X^\mu\partial_j X^\nu b_{\mu\nu}, \tag{11}$$

where $M = 0, 1, \ldots, 9, 11$ are spacetime indices, $\mu = 0, 1, \ldots, 5$ are 5-brane worldvolume indices, $\alpha = 0, 1, 2$ are membrane worldvolume indices and $i = 0, 1$ are indices on the boundary of the membrane.

The coupling of b to the boundary of the membrane ensures that the open-membrane action is invariant under the spacetime gauge transformations $\delta C = \mathrm{d}\Lambda$ provided that $\delta b = -f_5^*\Lambda$, where f_5 denotes the embedding of the 5-brane into spacetime. The 2-form b satisfies the 5-brane field equations. These are equivalent to a nonlinear self-duality condition on the following gauge invariant 3-form field strength of b:

$$\mathcal{H} = \mathrm{d}b + f_5^* C. \tag{12}$$

Here the last term is the pull-back of the spacetime 3-form potential to the 5-brane:

$$(f_5^* C)_{\mu\nu\rho} = \partial_\mu x^M \partial_\nu x^N \partial_\rho x^P C_{MNP}, \tag{13}$$

where $x^M(X^\mu)$ are local embedding functions satisfying the 5-brane equations of motion.

We shall consider backgrounds where $\mathcal{H}_{\mu\nu\rho}$ is constant. This is only consistent with (12) provided we require that the pull-back of the spacetime 4-form field strength $F = \mathrm{d}C$ to the 5-brane vanishes, i.e. $f_5^\star F = 0$. It is convenient to write $C = \tilde{C} + \mathrm{d}C_2$ with $f_5^\star \tilde{C} = 0$ and $f_5^* C_2 = c$. This enables us to rewrite the following bulk term as a boundary term:

$$\int_{M^3} f_2^\star C = \int_{\partial M^3} f_2^\star C_2 = \int_{\partial M^3} f_1^\star f_5^\star C_2 = \int_{\partial M^3} f_1^\star c, \tag{14}$$

where we have applied Stoke's theorem. Finally, since $f_5^\star C = \mathrm{d}c$ we have that $\mathcal{H} = \mathrm{d}(b + c)$ or

$$(b + c)_{\mu\nu} = \mathcal{H}_{\mu\nu\rho} X^\rho. \tag{15}$$

This enables us to rewrite the Wess–Zumino term as

$$S_{\mathrm{WZ}} = \tfrac{1}{3} \int_{\partial M^3} \mathrm{d}^2\sigma\, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho. \tag{16}$$

A complicating feature of the M5-brane is that the 3-form curvature $\mathcal{H}$ satisfies a nonlinear self-duality condition [4]:

$$\frac{\sqrt{-\det g}}{6} \epsilon_{\mu\nu\rho\sigma\lambda\tau} \mathcal{H}^{\sigma\lambda\tau} = \frac{1 + K}{2} (G^{-1})_\mu{}^\lambda \mathcal{H}_{\nu\rho\lambda}, \tag{17}$$

where $\epsilon^{012345} = 1$ and the scalar K and the tensor $G_{\mu\nu}$ are given by

$$K = \sqrt{1 + \frac{\ell_p^6}{24} \mathcal{H}^2}, \tag{18}$$

$$G_{\mu\nu} = \frac{1 + K}{2K} \left(g_{\mu\nu} + \frac{\ell_p^6}{4} \mathcal{H}^2_{\mu\nu} \right). \tag{19}$$

In [3] it was argued that the tensor $G_{\mu\nu}$ is the metric on the 5-brane seen by an open membrane in the presence of a background 3-form field strength $\mathcal{H}$. It is understood that in the above three equations the indices are contracted with the induced 5-brane metric:

$$g_{\mu\nu} = \partial_\mu x^M \partial_\nu x x^N \hat{g}_{MN}. \tag{20}$$

It will be useful to introduce a specific parametrization of the solutions of the self-duality condition (17) as follows[2]:

$$\mathcal{H}_{\mu\nu\rho} = \frac{h}{\sqrt{1+\ell_p^6 h^2}} \epsilon_{\alpha\beta\gamma} v_\mu^\alpha v_\nu^\beta v_\rho^\gamma + h\,\epsilon_{abc} u_\mu^a u_\nu^b u_\rho^c, \tag{21}$$

$$G_{\mu\nu} = \frac{\left(1+\sqrt{1+h^2\ell_p^6}\right)^2}{4}\left(\frac{1}{1+h^2\ell_p^6}\eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta + \delta_{ab} u_\mu^a u_\nu^b\right). \tag{22}$$

Here h is a real field of dimension (mass)^3 and (v_μ^α, u_μ^a), $\alpha = 0, 1, 2$, $a = 3, 4, 5$, are sechsbein fields in the nine-dimensional coset $SO(5,1)/SO(2,1)\times SO(3)$ satisfying

$$g^{\mu\nu} v_\mu^\alpha v_\nu^\beta = \eta^{\alpha\beta}, \qquad g^{\mu\nu} u_\mu^a v_\nu^\beta = 0, \qquad g^{\mu\nu} u_\mu^a u_\nu^b = \delta^{ab}, \tag{23}$$

$$g_{\mu\nu} = \eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta + \delta_{ab} u_\mu^a u_\nu^b. \tag{24}$$

3. Limits on M5

We will now consider a limit of the open-membrane/5-brane system with the main property that the boundary string that lives in the 5-brane is governed solely by the Wess–Zumino term (16). Compared with the case of a string ending on a Dp-brane we are faced with two problems.

(1) The decoupling limit must be consistent with the nonlinear self-duality condition (17).
(2) Since both $\tau_{\mathrm{M2}} \sim 1$ and $\tau_{\mathrm{M5}} \sim 1$ we cannot use the membrane as a probe to study the worldvolume geometry of the M5-brane.

In this paper we will discuss a particular limit that avoids these two problems. Other limits were discussed at this conference by Per Sundell. Problem (1) is circumvented by using the explicit solution for $\mathcal{H}$ given by (21) and (24). To take care of problem (2) we consider, instead of a flat background, a $D = 11$ background consisting of a stack of N parallel 5-branes, given by the solution

$$\mathrm{d}s^2(\hat{g}) = H^{-1/3}(\mathrm{d}x^\mu)^2 + H^{2/3}(\mathrm{d}y^m)^2, \qquad H = 1 + \frac{N_5\ell_p^3}{r^3}, \qquad F = N_5\epsilon_4, \tag{25}$$

where $\mu = 0, 1, \ldots, 5$; $m = 6, 7, 8, 9, 11$, N_5 is the number of stacked 5-branes and ϵ_4 is the volume form on the transverse S^4. We let the open membrane end on one of these 5-branes removed from the stack and placed at radius r_0. If $N_5 \gg 1$ and r_0 is small, then the interactions between the stack and the separated 5-brane effectively stiffens the latter so that the membrane can probe it without deforming it. Under these conditions the induced metric on the 5-brane (20) is given by

$$g_{\mu\nu} = H^{-1/3}(r_0)\eta_{\mu\nu}. \tag{26}$$

Moreover, from (25) it follows that the $D = 11$ background 4-form field strength satisfies $f_5^*F = 0$. From the discussion in section 2 it follows that we may consider an open-membrane action given by

$$S = \frac{1}{2(\ell_p)^2}\int_{M^3} \mathrm{d}^3\sigma\,\sqrt{-\det\gamma}\big(-H^{-1/3}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} - H^{2/3}\gamma^{\alpha\beta}\partial_\alpha Y^m\partial_\beta Y^n\delta_{mn} + \ell_p^2\big)$$
$$+N_5\int_{M^3} f_2^\star\tilde{C} + \frac{1}{3}\int_{\partial M^3}\mathrm{d}^2\sigma\,\mathcal{H}_{\mu\nu\rho}X^\mu\dot{X}^\nu X'^\rho, \tag{27}$$

[2] This parametrization has been derived independently by [5].

where the $D = 11$ background 3-form potential $\tilde{C}$ obeys $\mathrm{d}\tilde{C} = \epsilon_4$ and $f_5^*\tilde{C} = 0$ and the background 3-form field strength $\mathcal{H}_{\mu\nu\rho}$ on the 5-brane is constant.

We now propose the following decoupling limit obtained by taking $\epsilon \to 0$ such that

$$\ell_p \sim \epsilon\, \ell_p, \tag{28}$$

$$\frac{N_5}{r_0^3} \sim \epsilon^{-3\delta} \frac{N_5}{r_0^3}, \tag{29}$$

$$h \sim \epsilon^{-\lambda} h. \tag{30}$$

For simplicity we shall assume that $\delta > 1$ such that we may drop the 1 from the harmonic function in the metric (25). It then follows from (26) that the induced 5-brane metric and the sechsbein fields in (24) scale as

$$g_{\mu\nu} \sim \epsilon^{\delta-1} g_{\mu\nu}, \qquad u_\mu^a \sim \epsilon^{\frac{1}{2}(\delta-1)} u_\mu^a, \qquad v_\mu^\alpha \sim \epsilon^{\frac{1}{2}(\delta-1)} v_\mu^\alpha. \tag{31}$$

Furthermore, we assume that $\lambda \leqslant 3$. This implies that $h\ell_p^3$ remains finite which enables us to keep the 3-form field strength and the open-membrane metric (22) non-degenerate in the limit. Thus we find that the open-membrane action (27) scales as

$$S \sim \epsilon^{-\Delta} \Bigg[\frac{1}{2\ell_p^2} \int_{M^3} \mathrm{d}^3\sigma\, \sqrt{-\gamma} \big(-\epsilon^{\Delta+\delta-3} H^{-1/3} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \\ - \epsilon^{\Delta-2\delta} H^{2/3} \gamma^{\alpha\beta} \partial_\alpha Y^m \partial_\beta Y^n \delta_{mn} + \epsilon^\Delta \ell_p^2 \big) + \tfrac{1}{3} \int_{\partial M^3} \mathrm{d}^2\sigma\, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho \Bigg], \tag{32}$$

where we have defined

$$\Delta = \lambda - \tfrac{3}{2}(\delta - 1). \tag{33}$$

We now impose the following requirements for our decoupling limit (for a more detailed discussion, see [3]).

(a) All interactions on the M5-brane worldvolume must vanish.
(b) The bulk modes must decouple.
(c) The open-membrane metric must remain non-degenerate after taking the decoupling limit.

Given these assumptions we find the following restrictions on our parameters [3]:

$$\Delta + \tfrac{3}{2}(\delta - 1) \leqslant 3 < \Delta + \delta, \qquad \Delta < 2\delta, \qquad 1 < \delta < 3. \tag{34}$$

These conditions are solved by (Δ, δ) in a finite size region. For instance, $\delta = \frac{5}{3}$, $\lambda = 3$ and $\Delta = 2$ leads to a decoupled 5-brane theory in a background with a nonlinearly self-dual field strength, while $\delta = \frac{4}{3}$, $\lambda = 2$ and $\Delta = 2$ yields a linearly self-dual field strength.

A noteworthy feature is that (34) implies $\Delta > 0$, such that there is necessarily an overall scaling of the action in (32). Such a scaling was not required in the string case. A crucial difference between the string and the membrane is that only the string action (1) has a microscopic interpretation. On the other hand, the membrane action (10) should be seen as an effective action. One interpretation of the scaling (32) with $\Delta > 0$ is that actually we are taking a semiclassical limit.

Summarizing, in order to understand the geometry of the 5-brane worldvolume we are led to study the quantization of the Wess–Zumino action (16) with $\mathcal{H}$ constant.

4. Canonical analysis

We now will canonically quantize the action (16) with constant field strength $\mathcal{H}_{\mu\nu\rho}$. For convenience, we assume that the field strength can be diagonalized as follows [2]:

$$\mathcal{H}_{012} = -\frac{h}{\sqrt{1+\ell_p^6 h^2}}, \qquad \mathcal{H}_{345} = h, \tag{35}$$

where the dimensionless combination $h\ell_p^3$ is non-vanishing provided the decoupling limit (30) has been taken with $\lambda = 3$.

In the parametrization (35) the action (16) splits into two independent Lagrangians for the two sets of coordinates $X^{0,1,2}$ and $X^{3,4,5}$:

$$S = \frac{h}{3\sqrt{1+\ell_p^6 h^2}} \int_{\partial M^3} \mathrm{d}^2\sigma\, \epsilon_{\alpha\beta\gamma} X^\alpha \dot{X}^\beta X'^\gamma + \frac{h}{3} \int_{\partial M^3} \mathrm{d}^2\sigma\, \epsilon_{abc} X^a \dot{X}^b X'^c, \tag{36}$$

where $\alpha = 0, 1, 2$ and $a = 3, 4, 5$. The action is invariant under worldsheet reparametrizations:

$$\delta_\xi X^\alpha = \xi^i \partial_i X^\alpha, \qquad \delta_\eta X^a = \eta^i \partial_i X^a, \qquad i = 0, 1. \tag{37}$$

Note that, due to the absence of a worldsheet metric, there is no need to identify the vector fields ξ and η.

The equations of motion are

$$\epsilon_{\alpha\beta\gamma} \dot{X}^\beta X'^\gamma = 0, \qquad \epsilon_{abc} \dot{X}^b X'^c = 0. \tag{38}$$

Assume now that the string boundary inside the M5-brane has a non-compact extension in the time direction. In that case we can impose the gauge choice $X^0 = \tau$. Substituting this into the equations of motion we obtain

$$X'^\alpha = 0, \tag{39}$$

which means that the spatial extension of the string must be in the a direction. Assuming that $|\vec{X}'| \neq 0$ we obtain

$$\dot{X}^a = 0, \tag{40}$$

which implies additional Dirichlet conditions in the a directions.

Let us continue by analysing the phase space dynamics of the three coordinates $\vec{X} = (X^0, X^1, X^2)$. The canonical momenta are given by

$$\Pi_\alpha(\sigma) := \frac{\delta S}{\delta \dot{X}^\alpha(\sigma)} = -\tfrac{1}{3} h \epsilon_{\alpha\beta\gamma} X^\beta X'^\gamma, \tag{41}$$

indicating that there are three primary constraints $\phi_\alpha(\sigma)$:

$$\phi_a := \Pi_a + \tfrac{1}{3} h \epsilon_{abc} X^b X'^c \approx 0. \tag{42}$$

The non-trivial canonical Poisson brackets are

$$\{X^a(\sigma), \Pi_b(\sigma')\} = \delta^a_b\, \delta(\sigma - \sigma') \tag{43}$$

and the non-zero Hamiltonian is given by

$$H = \int \mathrm{d}\sigma\, \lambda^a(\sigma) \phi_a(\sigma), \tag{44}$$

where $\lambda^a(\sigma)$ are three Lagrange multipliers. To proceed with the canonical analysis we study the consistency conditions

$$\dot{\phi}_a(\sigma) = \lambda^b(\sigma) M_{ba}(\sigma) \approx 0, \tag{45}$$

where

$$\{\phi_a(\sigma), \phi_b(\sigma')\} = M_{ab}(\sigma)\delta(\sigma - \sigma'), \qquad M_{ab} = h\,\epsilon_{abc} X'^c. \tag{46}$$

Note that in the α space we can impose $X^0 = \tau$ and, via the equations of motion, $X'^\alpha = 0$. This implies that $M_{\alpha\beta} = 0$. In other words, the three primary constraints $\phi_\alpha(\sigma)$ are all first class.

In contrast, let us now consider the canonical analysis of the three Euclidean coordinates $\vec{X} = (X^3, X^4, X^5)$. A similar analysis as above leads to the same result except that in this case we have assumed that $|\vec{X}'| \neq 0$ and therefore

$$M_{ab} X'^b = 0. \tag{47}$$

The matrix M_{ab} is thus non-degenerate in the two-dimensional subspace orthogonal to $\vec{X}'$. It is convenient to introduce a projection onto this subspace as follows ($I = 1, 2$):

$$P_I{}^a(\sigma) P_J{}^b(\sigma)\delta_{ab} = \delta_{IJ}, \tag{48}$$

$$\delta^{IJ} P_I{}^a(\sigma) P_J{}^b(\sigma) = \delta^{ab} - \frac{X'^a X'^b}{|\vec{X}'|^2}, \tag{49}$$

$$\epsilon^{IJ} P_I{}^a(\sigma) P_J{}^b(\sigma) = \frac{\epsilon^{abc} X'^c}{|\vec{X}'|}. \tag{50}$$

The three constraints ϕ_a now split into the two second-class constraints

$$\chi_I := P_I{}^a \phi_a, \tag{51}$$

with the now non-degenerate matrix

$$\{\chi_I(\sigma), \chi_J(\sigma')\} := M_{IJ}(\sigma)\delta(\sigma - \sigma'), \qquad M_{IJ} = P_I{}^a P_J{}^b M_{ab}, \tag{52}$$

and one first-class constraint

$$\phi := X'^a \phi_a \equiv X'^a \Pi_a, \tag{53}$$

which acts as the generator of σ reparametrizations.

The presence of the two second-class constraints leads to a non-trivial Dirac bracket between the X^a coordinates given by

$$[X^a(\sigma), X^b(\sigma')]^D = -\frac{1}{h} \frac{\epsilon^{abc} X'^c(\sigma)}{|\vec{X}'(\sigma)|^2} \delta(\sigma - \sigma'). \tag{54}$$

The conclusion is that the membrane probe sees a so-called non-commutative loop space geometry in the a directions of the M5-brane worldvolume.

5. Non-commutative loop space

The main conclusion of this paper is that, whereas D-branes lead to a non-commutative geometry of points, the M5-brane seems to lead to a non-commutative geometry of loops. To the best of our knowledge, such a non-commutative loop space geometry has not been considered before in the literature.

As a historical note, it is perhaps of interest to note that, whereas the idea of lightlike integrability applied to a superspace geometry naturally leads to the superspace constraints of Yang–Mills [6], the same idea when applied to a loop superspace geometry leads to the constraints of supergravity coupled to Yang–Mills [7]. In the latter work the definition of a loop space covariant derivative plays a central role. The gauge field part of this covariant derivative is given by the pull-back of the self-dual antisymmetric tensor, i.e.

$$\mathcal{D}_\mu(\sigma) = \frac{\delta}{\delta X^\mu(\sigma)} + b_{\nu\mu} X'^{\nu}. \tag{55}$$

Through this paper we are naturally led to consider a non-commutative version of loop superspace. One of the open questions is how to exactly construct a covariant derivative corresponding to such a non-commutative loop space. It suggests that this problem is related to the problem of how to construct a field theory for a set of $D = 6$ (2, 0) non-Abelian tensor multiplets. The analogy is as follows. On the one hand, in the non-commutative case, one must replace the term $b_{\nu\mu} X'^{\nu}$, present in the covariant derivative, by some non-commutative generalization with

$$\{X^\mu, X^\nu\} \neq 0. \tag{56}$$

On the other hand, in the non-Abelian case, one must replace this term by some non-Abelian generalization, i.e. $b^I_{\nu\mu} X'^{\nu} T^I$ with

$$[T^I, T^J] \neq 0. \tag{57}$$

More generally, the suggestion is that, in order to describe a set of $D = 6$ (2, 0) tensor multiplets, it will not suffice to work with a local field theory but instead, one should work with a non-local loop space where in the covariant derivative one makes the replacement:

$$b_{\nu\mu}(X) X'^{\nu} \quad \Longrightarrow \quad A_\mu\big(X(\sigma)\big). \tag{58}$$

The antisymmetric tensor is just one of the many components of the gauge field $A_\mu\big(X(\sigma)\big)$. In this way one would also circumvent the no-go theorem of [8]. It would be of interest to investigate these issues in more detail.

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